Journal of Statistical Physics, Vol. 5, No. 3, 1972

Action–Angle Variables in Quantum Statistical Mechanics

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Received September 9, 1971

Utilizing the facts (i) that the number of particles in the many-boson system is conserved and (ii) that the Hamiltonian is Hermitian, a new set of variables comprising "action" and "angle" variables has been introduced. These variables are conjugate in the "mean" and provide a rigorous approach to introducing phase variables for "total-number-conserving many-boson systems."

KEY WORDS: Coherent states; number conservation; action–angle variables; conjugate property.

1. INTRODUCTION

Referring to the second-quantization formalism of quantum mechanics, let a_K and a_K^+ be the operators which respectively annihilate or create a boson of momentum **K**. One defines a coherent state[1] $|\,\alpha_K\rangle$ satisfying

$$a_K\,|\,\alpha_K\rangle = \alpha_K\,|\,\alpha_K\rangle \tag{1}$$

where α_K can be any complex number. It is well known[1–3] that these states $|\,\alpha_K\rangle$ do not form an orthogonal set, but they are complete, and, in fact, they are overcomplete. A beautiful review of these states is given by Glauber.[1]

These states have been extensively used in optics by Glauber and in describing the irreversible processes in anharmonic crystals by Carruthers and Dy.[4] It was naturally felt that they could be used for interacting Boson

Lecture given at the Saha Institute of Nuclear Physics, Calcutta, June 1971.

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systems, too.[5] However, all such attempts[5,6] suffer from the fault that they formally confine themselves to a mathematical description of the density matrix in terms of coherent states, forgetting the physical fact that the many-boson system is basically different from the system of phonons, due to the finite mass of the particles, and that this restriction, expressed in terms of the conservation of the number of particles, should be explicity incorporated in the description of the physical system. In what follows, we shall see how this can be done. We shall further see that it is this restriction that allows one to define phase variable ϕ_K which is conjugate in the mean to the variable N_K, the number of bosons in the momentum state K.

2. DENSITY MATRIX

Let a system of bosons be described by a Hamiltonian (in the second-quantized formalism)

$$H = H_0 + V, \qquad \text{where} \quad H_0 = \sum_K \omega_K a_K^+ a_K\,, \quad V = V(\{a_K^+, a_K\}) \tag{2}$$

a_K and a_K^+ being the annihilation and the creation operators, respectively, for a boson in the momentum state K. Here, $V(\{a_K^+, a_K\})$ is the interaction potential, which can be written in a convergent series of the form

$$V(\{a_K^+, a_K\}) = \sum_{\{n_K, m_K\}} V_{\{n_K, m_K\}}(a_K^+)^{n_K}(a_K)^{m_K} \tag{3}$$

Let us introduce the coherent states by

$$\begin{aligned} a_K \mid \alpha_K\rangle &= \alpha_K \mid \alpha_K\rangle \\ \langle\alpha_K \mid a_K^+ &= \alpha_K^*\langle\alpha_K \mid \end{aligned} \tag{4}$$

where α_K is a complex number. We also write (as usual)

$$\alpha_K = (J_K/\hbar)^{1/2} e^{i\phi_K} \tag{5}$$

Let ρ be the density matrix of the system satisfying the von Newmann equation

$$i\hbar\, \partial\rho/\partial t = [H, \rho] \tag{6}$$

It can be easily shown by using the properties of complex variables that, given any observable

$$A(\{a_K^+, a_K\}) = \sum_{\{n_K, m_K\}} A_{\{n_K, m_K\}}(a_K^+)^{n_K}(a_K)^{m_K} \tag{7}$$

(a convergent series), the mean value of A can be written as [2,4,5]

$$\begin{aligned}\langle A\rangle &= \mathrm{Tr}\, A\rho \\ &= \int \left[\exp\left(-\sum_K |\alpha_K|^2\right)\right] A(\{\alpha_K{}^*, \partial/\partial\alpha_K{}^*\}) \\ &\quad \times \left[\exp\left(\sum_K |\alpha_K|^2\right)\right] \rho(\{\alpha_K{}^*, \alpha_K\}) \prod_K d^2\alpha_K \end{aligned} \tag{8}$$

where

$$\rho(\{\alpha_K{}^*, \alpha_K\}) = \langle\{\alpha_K\} | \rho | \{\alpha_K\}\rangle$$

with

$$|\{\alpha_K\}\rangle = \prod_K |\alpha_K\rangle \tag{9}$$

This shows that it is enough to know the function $\rho(\{\alpha_K{}^*, \alpha_K\})$, which gives the complete statistical description of the system. This is possible because the functions $\rho(\{\beta_K{}^*, \alpha_K\})$ formed out of the nondiagonal elements of the density matrix in terms of the coherent states can be obtained as boundary values by analytic continuation of the diagonal elements (α_K and $\alpha_K{}^*$ are treated as independent).[1] As noted by many,[2,4,5] $\rho(\{\alpha_K{}^*, \alpha_K\})$ satisfies the equations [due to (6)]

$$\begin{aligned} i\hbar\, \partial\tilde\rho(\{\alpha_K{}^*, \alpha_K\})/\partial t &= [H(\{\alpha_K{}^*, \partial/\partial\alpha_K{}^*\}) - H^+(\{\alpha_K, \partial/\partial\alpha_K\})]\, \tilde\rho(\{\alpha_K{}^*, \alpha_K\}) \\ \tilde\rho(\{\alpha_K{}^*, \alpha_K\}) &= \rho \exp\left(\sum_K |\alpha_K|^2\right) \end{aligned} \tag{10}$$

where H^+ is the Hermitian conjugate of H.

3. NUMBER CONSERVATION AND ACTION–ANGLE VARIABLES

Let $|n_K\rangle$ be the state vectors in the Fock space, i.e., the space spanned by the eigenvectors of the operator $a_K{}^+a_K$,

$$a_K{}^+a_K |n_K\rangle = n_K |n_K\rangle, \qquad a_K |n_K\rangle = \sqrt{n_K}\, |n_K - 1\rangle \tag{11}$$

$$a_K{}^+ |n_K\rangle = \sqrt{n_K}\, |n_K + 1\rangle \tag{12}$$

The coherent state $|\alpha_K\rangle$ can be expanded in terms of the complete orthonormal set $\{|n_K\rangle\}$. The expansion is

$$|\alpha_K\rangle = [\exp(-\tfrac{1}{2} |\alpha_K|^2)] \sum_{n_K} [\alpha_K^{n_K}/(n_K!)^{1/2}]\, |n_K\rangle \tag{13}$$

The density function $\rho(\{\alpha_K^*, \alpha_K\}) = \langle\{\alpha_K\} \mid \rho \mid \{\alpha_K\}\rangle$ can therefore be written as

$$\rho(\{\alpha_K^*, \alpha_K\}) = \sum_{\{n_K, m_K\}} \left(\exp - \sum_K \mid \alpha_K \mid^2\right) \times \prod_K [(\alpha_K^*)^{m_K} \alpha_K^{n_K}/(n_K!\, m_K!)^{1/2}] \langle\{m_K\} \mid \rho \mid \{n_K\}\rangle \quad (14)$$

Let us introduce the variables

$$n_K + m_K = 2N_K \qquad n_K - m_K = \nu_K \quad (15)$$

and define

$$A_{\{\nu\}_K}(\{N_K\}) = \langle\{N_K - \tfrac{1}{2}\nu_K\} \mid A \mid \{N_K + \tfrac{1}{2}\nu_K\}\rangle \quad (16)$$

where A is any operator.

From (1) and (5), we have

$$\rho(\{J_K, \phi_K\}) = \sum_{\{\nu_K\}} \rho_{\{\nu_K\}}(\{J_K\}) \exp\left(i \sum_K \nu_K \phi_K\right) \quad (17)$$

where

$$\sum_K \nu_K = 0 \quad (18)$$

due to number conservation and

$$\rho_{\{\nu_K\}}(\{J_K\}) = \sum_{\{N_K\}} \left[\exp\left(- \sum_K J_K/\hbar\right)\right] \times \prod_K \{(J_K/\hbar)^{N_K}/[(N_K + \tfrac{1}{2}\nu_K)!(N_K - \tfrac{1}{2}\nu_K)!]^{1/2}\} \rho_{\{\nu_K\}}(\{N_K\}) \quad (19)$$

$\rho_{\{\nu_K\}}(\{N_K\})$ satisfies the equation[8]

$$\begin{aligned} i\hbar\, \partial\rho_{\{\nu_K\}}(\{N_K\})/\partial t &= \sum_{\{\nu_K'\}} [H_{\{\nu_K - \nu_K'\}}(\{N_K + \tfrac{1}{2}\nu_{K'}\})\, \rho_{\{\nu_K'\}}(\{N_K - \tfrac{1}{2}\nu_K + \tfrac{1}{2}\nu_{K'}\}) \\ &\quad - H_{\{\nu_K - \nu_K'\}}(\{N_K - \tfrac{1}{2}\nu_{K'}\})\, \rho_{\{\nu_K'\}}(\{N_K + \tfrac{1}{2}(\nu_K - \nu_{K'})\})] \\ &= \sum_{\{\nu_K'\}} \{[\exp(\tfrac{1}{2}\nu_{K'}\, \partial/\partial N_K)]\, H_{\{\nu_K - \nu_K'\}}(\{N_K\})[\exp(- \tfrac{1}{2}\nu_K\, \partial/\partial N_K)] \\ &\quad - [\exp(- \tfrac{1}{2}\nu_{K'}\, \partial/\partial N_K)]\, H_{\{\nu_K - \nu_K'\}}(\{N_K\})[\exp(\tfrac{1}{2}\nu_K\, \partial/\partial N_K)]\}\, \rho_{\{\nu_K'\}}(\{N_K\}) \end{aligned} \quad (20)$$

Equation (18) is the crucial one in our discussion, incorporating the fact that the bosons we are dealing with are finite-mass particles. Since the set

$\{\exp i\sum_K \nu_K\phi_K\}$ is complete and orthonormal, (17) defines the unique Fourier series of $\rho(\{J_K, \phi_K\})$. The variables J_K and ϕ_K such that the Fourier series (17) of $\rho(\{J_K, \phi_K\})$ is restricted to the relation (18) will be defined as the "action" and the "angle" variables of the quantum statistical system. These variables are therefore distinct from those of Carruthers *et al.*[2] and others[5,6] who make use of the "amplitudes" and "phases" of α's without any physical restrictions.

4. CONJUGATE PROPERTY OF J_K, N_K, AND ϕ_K

Now, from (17) and (19), we have

$$\langle J_K\rangle = \int_0^\infty J_K\rho_0(\{J_K\}) \prod_K (dJ_K/\hbar) = [\langle N_K\rangle + 1]\hbar$$

where

$$\langle N_K\rangle = \sum_{\{N_K\}} N_K\rho_0(\{N_K\})$$

$$\int_0^\infty \rho_0(\{J_K\}) \prod_k dJ_K = \sum_{\{N_K\}} \rho_0(\{N_K\}) = 1 \tag{21}$$

Let us define

$$\mathscr{A}(\{J_K, \phi_K\}) = \langle\{\alpha_K\} \mid A \mid \{\alpha_K\}\rangle \tag{22a}$$

expressed in terms of $\{J_K, \phi_K\}$ by (5),

$$A(\{N_K, \phi_K\}) = \sum_{\{\nu_K\}} A_{\{\nu_K\}}(\{N_K\}) \exp i \sum_K \nu_K\phi_K \tag{22b}$$

where A is any operator.

Let us further define

$$\langle J_K\rangle = \int_0^\infty \int_0^{2\pi} J_K\rho(\{J_K, \phi_K\}) \prod_K (dJ_K/\hbar)(d\phi_K/2\pi) \tag{23}$$

$$\langle N_K\rangle = \sum_K \int N_K\rho(\{N_K, \phi_K\}) \prod_K (d\phi_K/2\pi) = \sum_{\{N_K\}} N_K\rho_0(\{N_K\}) \tag{24}$$

$$\langle \phi_K\rangle = \int_0^{2\pi} \phi_K\rho(\{J_K, \phi_K\}) \prod_K (dJ_K\hbar)/(d\phi_K/2\pi) = \sum_K \int_0^{2\pi} \phi_K\rho(\{N_K, \phi_K\}) \prod_K (d\phi_K/2\pi) \tag{25}$$

Observation 1. In definition (24), one must note carefully that a single ϕ_K picks out a single ν_K in the integrals and all the other $\nu_{K'}$ $(K' \neq K)$ must vanish. Now, due to number conservation, $\sum_K \nu_K = 0$; hence this single ν_K must tend to zero. It is in this sense that all the "means" will be defined so that *after* the integration is over, one must take the limit of $\nu_K \to 0$ to impose the physical restriction of number conservation. This fact must be accepted as a *convention.*

Proposition 1. ϕ_K and N_K are conjugate in the "mean" and satisfy the equations

$$d\langle N_K\rangle/dt = \langle \dot{N}_K\rangle = (1/\hbar)\langle \partial H(\{N_K, \phi_K\})/\partial\phi_K\rangle \tag{26a}$$

$$d\langle \phi_K\rangle/dt = \langle \dot{\phi}_K\rangle = -(1/\hbar)\langle \partial H(\{N_K, \phi_K\})/\partial N_K\rangle \tag{26b}$$

Proof. Using (24) and (17),

$$\begin{aligned}
d\langle N_K\rangle/dt &= (1/\hbar)\,\langle J_K\rangle/dt \\
&= (1/\hbar)\int_0^\infty \int_0^{2\pi} J_K[\partial\rho(\{J_K, \phi_K\})/\partial t] \\
&\quad \times \prod_k (dJ_k/\hbar)(d\phi_k/2\pi) \\
&= \sum_{N_K} N_K[\partial\rho_{\{0\}}(\{N_K\})/\partial t]
\end{aligned} \tag{27}$$

[using Eq. (20) and observation 1]

$$\begin{aligned}
&= (1/i\hbar)\sum_{\nu_K}\sum_{N_K} N_K[\langle\{N_K + \nu_K\} \mid H \mid \{N_K\}\rangle\rho_{\{\nu_K\}}(\{N_K + \tfrac{1}{2}\nu_K\}) \\
&\quad - \langle\{N_K \mid H \mid \{N_K - \nu_K\}\rangle\, \rho_{\{\nu_K\}}(\{N_K - \tfrac{1}{2}\nu_K\}) \\
&= (1/i\hbar)\sum_{\nu_K}\sum_{N_K'} \{(N_K' - \tfrac{1}{2}\nu_K)\langle\{N_K' + \tfrac{1}{2}\nu_K\} \mid H \mid \{N_K' - \tfrac{1}{2}\nu_K\}\rangle \\
&\quad - (N_K' + \tfrac{1}{2}\nu_K)\langle\{N_K' + \tfrac{1}{2}\nu_K\} \mid H \mid \{N_K' - \tfrac{1}{2}\nu_K\}\rangle\}\, \rho_{\{\nu_K\}}(\{N_K'\}) \\
&= -(1/i\hbar)\sum_{\nu_K}\sum_{N_K} \nu_K\langle\{N_K' + \tfrac{1}{2}\nu_K\} \mid H \mid \{N_K' - \tfrac{1}{2}\nu_K\}\rangle\, \rho_{\nu_K}\{N_K'\} \\
&= (i/\hbar)\sum_{\nu_K}\sum_{N_K} \nu_K H_{-\nu_K}(\{N_K\})\, \rho_{\{\nu_K\}}(\{\kappa'\}) \\
&= (1/\hbar)\langle\partial H^{+}(\{N_K, \phi_K\})/\partial\phi_K\rangle
\end{aligned}$$

(the Hermitian conjugate refers to the Fock space). Since H is Hermitian,

$$\langle \dot{N}_K \rangle = \frac{1}{\hbar} \left\langle \frac{\partial H(\{N_K, \phi_K\})}{\partial \phi_K} \right\rangle$$

$$\frac{d}{dt} \langle \phi_K \rangle = \int_0^\infty \int_0^{2\pi} \phi_K \frac{\partial \rho(\{J_K, \phi_K\})}{\partial t} \prod_K \left(\frac{dJ_K}{\hbar}\right)\left(\frac{d\phi_K}{2\pi}\right)$$

$$= \sum_{\{\nu_K\}} \sum_{\{N_K\}} \frac{\partial \rho_{\{\nu_N\}}(\{N_K\})}{\partial t} \frac{N_K!}{[(N_K - \frac{1}{2}\nu_K)!\,(N_K + \frac{1}{2}\nu_K)!]^{1/2}}$$

$$\left[\pi \delta_{\nu_K,0} + \frac{1}{i\nu_K}\right] \tag{28}$$

Noting that $\sum_K \nu_K = 0$ and

$$\sum_{\{N_K\}} [\partial \rho_{\{0\}}(\{N_K\})/\partial t] = 0 \qquad (\therefore \quad \mathrm{Tr}\, \rho = 1)$$

$$d\langle \phi_K \rangle / dt = \lim_{\nu_K \to 0} \sum_{\{\nu_K'\}} \sum_{\nu_K} \sum_{\{N_K\}} (1/i\nu_K)(1/i\hbar)$$

$$\times [H_{\{\nu_K - \nu_K'\}}(\{N_K' + \tfrac{1}{2}\nu_K\}) - H_{\{\nu_K - \nu_K'\}}(\{N_K' - \tfrac{1}{2}\nu_K\})]\, \rho_{\{\nu_K'\}}(\{N\}_K'$$

$$= -(1/\hbar) \sum_{\{N_K\}} \sum_{\{\nu_K'\}} [\partial H_{\{-\nu_K'\}}(\{_K\})/\partial N_K]\, \rho_{\{\nu_K'\}}(\{N_K'\})$$

$$= -(1/\hbar) \langle \partial H^+(\{N_K, \phi_K\})/\partial N_K \rangle$$

(the Hermitian conjugate refers to the Fock space). Since H is Hermitian,

$$d\langle \phi_K \rangle / dt = -(1/\hbar) \langle \partial H(\{N_K, \phi_K\})/\partial N_K \rangle \tag{29}$$

Proposition 2. From (22a), (23), (25), (28), and (29), it also follows that

$$\begin{aligned} \langle \dot{J}_K \rangle &= \langle \partial H(\{J_K, \phi_K\})/\partial \phi_K \rangle \\ \langle \dot{\phi}_K \rangle &= -\langle \partial \mathscr{H}(\{J_K, \phi_K\})/\partial J_K \rangle \end{aligned} \tag{30}$$

Instead of giving a formal proof of the relations (30), we shall verify them for the many-boson Hamiltonian (in familiar notation)

$$H = \sum_K \omega_K a_K^+ a_K + (1/2\Omega) \sum_{1234} \langle 12 \mid V \mid 34 \rangle\, a_{K_1}^+ a_{K_2}^+ a_{K_3} a_{K_4} \tag{31}$$

Then $\mathscr{H}(\{J, \phi\})$ becomes [according to (22a)]

$$\mathscr{H}(\{J_K, \phi_K\}) = \sum_K \omega_K(J_K/\hbar) + (1/2\Omega\hbar^2) \sum_{1234} \langle 12 \mid V \mid 34\rangle \times (J_1J_2J_3J_4)^{1/2} e^{-i(\phi_1+\phi_2-\phi_3-\phi_4)}$$

$$= \sum_{\nu_1,\nu_2} H_{\nu_1\nu_2}(J_1, J_2) e^{i(\nu_1\phi_1+\nu_2\phi_2+\cdots)}$$

where

$$H_{\{\nu\}}(\{J\}) = \sum_K \omega_K(J_K/\hbar) \prod_q \delta_{\nu_q,0} + (1/2\Omega\hbar^2) \sum_{1234} \langle 12 \mid V \mid 34\rangle(J_1J_2J_3J_4)^{1/2} \times \delta_{\nu_1,1}\delta_{\nu_2,1}\delta_{\nu_3,-1}\delta_{\nu_4,-1} \prod_{q\neq 1,2,3,4} \delta_{\nu_q,0} \tag{32}$$

Let

$$\mathscr{C}_{\nu_K,N_K} = [(N_K - \tfrac{1}{2}\nu_K)!\,(N_K + \tfrac{1}{2}\nu_K)!]^{1/2}$$

so that

$$\mathscr{C}_{1,N} = [(N + \tfrac{1}{2})!\,(N - \tfrac{1}{2})!]^{1/2}$$

From (16) and (31),

$$H_{\{\nu\}}(\{N_K\}) = \sum_K \omega_K N_K \prod_q \delta_{\nu_2,0} + (1/2\Omega) \sum_{1234} \langle 12 \mid V \mid 34\rangle(N_3 + \tfrac{1}{2})^{1/2}(N_4 + \tfrac{1}{2})^{1/2} \times (N_1 + \tfrac{1}{2})^{1/2}(N_2 + \tfrac{1}{2})^{1/2} \delta_{\nu_3,-1}\delta_{\nu_4,-1}\delta_{\nu_1,1}\delta_{\nu_2,1} \prod_{q\neq 1,2,3,4} \delta_{\nu_q,0} \tag{33}$$

and

$$\langle \partial H(\{J_K, \phi_K\})/\partial\phi_K\rangle = \sum_{\{\nu_K\}} i\nu_K \sum_{\{N_K\}} \left[\rho_{\{\nu_K\}}(\{N_K\})\Big/\prod_q \mathscr{C}_{\nu_q,N_q}\right] \times \int_0^\infty \left[\exp - \sum (J_q/\hbar)\right] \prod_q J_q^{N_2} H_{\{-\nu_K\}}(\{J_K\})\, dJ_q$$

$$= i \sum_{\{\nu_K\}} \nu_K \sum_{\{N_K\}} \rho_{\{\nu_K\}}(\{N_K\}) \Big\{ \sum_K (N_K + 1) \prod_q \delta_{\nu_q,0}$$

$$+ \tfrac{1}{2} \sum_{1234} \langle 12 \mid V \mid 34 \rangle (N_1 + \tfrac{1}{2})^{1/2} (N_2 + \tfrac{1}{2})^{1/2}$$

$$\times (N_3 + \tfrac{1}{2})^{1/2} (N_4 + \tfrac{1}{2})^{1/2} \, \delta_{\nu_1,-1} \delta_{2\nu,-1} \delta_{\nu_3,1} \delta_{\nu_4,1} \prod_{q \neq 1,2,3,4} \delta_{\nu_q,0} \Big\}$$

$$= i \sum_{\{\nu_K\}} \gamma_K \sum_{\{N_K\}} \rho_{\{\nu_K\}}(\{N_K\}) \Big\{ H_{\{-\nu_K\}}(\{N_K\}) + \sum_K \omega_K \prod_q \delta_{\nu_q,0} \Big\}$$

$$= \sum_{\{\nu_K\}} \nu_K \sum_{\{N_K\}} \rho_{\{\nu_K\}}(\{\kappa\}) \, H_{\{-\nu_K\}}(\{N_K\})$$

$$= \hbar \, d\langle N_K \rangle / dt = d \langle J_K \rangle / dt \tag{34}$$

Again,

$$\langle \partial \mathscr{H}(\{J_K, \phi_K\}) / \partial J_K \rangle$$

$$= \int_0^\infty [\partial H(\{J_K, \phi_K\}) / \partial J_K] \, \rho(\{J_K, \phi_K\}) \prod_K dJ_K (\phi_K / 2\pi)$$

$$= \sum_{\{\nu_K\}} \sum_{\{N_K\}} \Big[\rho_{\{\nu_K\}}(\{N_K\}) \Big/ \prod_K \mathscr{C}_{\nu_K, N_K} \Big]$$

$$\times \int_0^\infty d\{J_K\} [\partial H_{\{-\nu_K\}}(\{J_K\}) / \partial J_K] \exp - \sum_K (J_K / \hbar)$$

$$= \sum_{\{\nu_K\}} \sum_{\{N_K\}} \Big[(\omega_K / \hbar) \prod_q \delta_{\nu_q,0} \mathscr{C}_{\nu_q, N_q}$$

$$+ (1/\Omega \hbar^2) \sum_{234} \langle k2 \mid V \mid 34 \rangle (N_2 + \tfrac{1}{2})^{1/2} (N_3 + \tfrac{1}{2})^{1/2}$$

$$\times (N_4 + \tfrac{1}{2})^{1/2} (N_K + \tfrac{1}{2})^{-1/2} \delta_{\nu_K,-1} \delta_{\nu_2,-1} \delta_{\nu_3,1} \delta_{\nu_4,1}$$

$$\times \mathscr{C}_{\nu_K, N_K} \mathscr{C}_{\nu_2, N_2} \mathscr{C}_{\nu_3, N_3} \mathscr{C}_{\nu_4, N_4} \prod_{q \neq k,2,3,4} \delta_{\nu_q,} \mathscr{C}_{\nu_q, N_q}$$

$$+ (1/\Omega \hbar^2) \sum_{234} \langle 34 \mid V \mid k2 \rangle (N_2 + \tfrac{1}{2})^{1/2} (N_3 + \tfrac{1}{2})^{1/2}$$

$$\times (N_4 + \tfrac{1}{2})^{1/2} (N_K + \tfrac{1}{2})^{-1/2} \delta_{K,1} \delta_{\nu_2,1} \delta_{\nu_3,-1} \delta_{\nu_4,-1} \mathscr{C}_{\nu_K, N_K} \mathscr{C}_{\nu_2, N_2}$$

$$\times \mathscr{C}_{\nu_3, N_3} \mathscr{C}_{\nu_4, N_4} \prod_{q \neq k,2,3,4} \delta_{\nu_q,0} \mathscr{C}_{\nu_q, N_q} \Big\}$$

$$= \sum_{\{\nu_K\}} \sum_{\{N_K\}} \rho_{\{\nu_K\}}(\{N_K\}) \, \partial H_{\{-\nu_K\}}(\{N_K\}) / \partial N_K$$

$$= (1/\hbar) \langle \partial H(\{N_K, \phi_K\}) / \partial N_K \rangle = - \, d \langle \phi_K \rangle / dt \tag{35}$$

Discussion. The necessity of using phase variables in quantum statistics has been very much felt during recent years. In fact, phenomena like the tunneling effect in superconductors or similar effects in superfluid helium strongly suggest that these systems could be well-described in terms of "number" and "phase" as microscopic variables conjugate to one another. As has been observed by many,[2] a "phase" operator conjugate to the "number" operator does not exist. This problem has been approached mainly along the following lines.

1. One assumes without worrying about rigor that there exists a phase operator Φ conjugate to the "number" operator N, argument being that the error made is small for large systems. This is the point of view adopted by Anderson.[9]

2. Instead of taking phase as the operator conjugate to N, one [2,10] uses a periodic function of the phase operator Φ.

3. One[6] uses coherent states which automatically introduce phases. But the states, being overcomplete, one uses a quasiprobability function.

The problem with the first approach has been discussed by Carruthers and Nieto.[2] Since one does not know exactly what error one is making in this assumption, one cannot use it in a rigorous formulation. Besides, no many-body Hamiltonian can be expressed in terms of only the number operator and (say) "phase" operators. So, practically, one cannot make use of them.

The second approach does not give a unique description. One does not know why one should use one or the other of the periodic functions of the "phase." Besides, the "phase" itself has a physical meaning, given a periodic function of the "phase" does not determine the phase.

The third approach suffers from the fact that the quasiprobability functions cannot describe strictly quantum mechanical systems. One does not know what connection they have with the physical distributions. In fact, it is misleading to use them for such systems as superfluids where zero-point quantum fluctuations play the most important role.

Though we have used coherent states to introduce the "action–angle" variables, our approach differs from the earlier ones in the combined use of the following facts: (1) The function $\rho(\{\alpha_K^*, \alpha_K\})$ is the relevant distribution for a many-boson system. (2) Number conservation should be explicitly used in the formulation.

Equations (26) and (30) are exact and are directly applicable to many-boson systems. The physical meaning of the "phase" is clearly brought in by these equations, giving that the time derivative of the "mean" value of the phase ϕ_K equals the negative of the local chemical potential $\mu_K = \langle \partial H / \partial N_K \rangle$ for a number-conserving boson system. We have used them in the discussion of tunneling effects in superconductors and for deriving exact hydrodynamic equations for many-boson systems. We shall not discuss them here, and the relevant references[14,15] should be consulted.

ACKNOWLEDGMENTS

My thanks are due to Dr. I. Rama Rao and Mr. S. Srivastava for raising many questions regarding the contents of this manuscript and thus helping to clarify many points.

REFERENCES

1. R. J. Glauber, in *Fundamental Problems in Statistical Mechanics* (Interscience, New York, 1968), Ed. by E. G. D. Cohen, and references therein.
2. P. Carruthers and M. M. Nieto, *Rev. Mod. Phys.* **40**:411 (1968).
3. K. E. Cahill, *Phys. Rev.* **138**:B1566 (1965).
4. P. Carruthers and K. S. Dy, *Phys. Rev.* **147**:214 (1966).
5. J. S. Langer, *Phys. Rev.* **167**:183 (1968); **184**:219 (1969).
6. F. W. Cummings and J. R. Johnston, *Phys. Rev.* **15**:105 (1966).
7. S. Bochner, W. T. Martin, and W. Ted, *Several Complex Variables* (Princeton University Press, Princeton, N. J., 1948).
8. P. Résibois, *Physica* **27**:541 (1961).
9. P. W. Anderson, *Phys. Rev.* **112**:1900 (1958).
10. M. M. Nieto, *Phys. Rev.* **167**:416 (1968).
11. M. M. Nieto, *Nuovo Cimento* **62B**:95 (1969).
12. A. C. Biswas, *Physica* **48**:123 (1970).
13. A. C. Biswas, *Phys. Letters* **30A**:296 (1969).
14. A. C. Biswas and C. S. Warke, *Phys. Rev.* **2A**:2568 (1970).
15. A. C. Biswas and I. Rama Rao, *Physica* **493** (1971).